Diffusion Tensor MR Imaging of Principal Directions: A Tensor Tomography Approach

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Abstract

This paper investigates a novel approach of reconstructing the principal directions of a diffusion tensor field directly from magnetic resonance imaging (MRI) data using a tensor tomography data acquisition approach. Because tensor eigenvalues are assumed to be known, the reconstruction of principal directions requires less measurements than the reconstruction of the full tensor field. The tensor tomography data acquisition method (rotated diffusion gradients) leads to a unique reconstruction of principal directions while the conventional MRI acquisition technique (stationary diffusion gradients) leads to an ambiguous reconstruction of principal directions if the same number of measurements are used. A computer generated phantom was used to simulate the diffusion tensor field in the mid-ventricular region of the myocardium. The diffusion model in this study depends upon the fiber structure of the myocardium. An iterative algorithm was used in the reconstruction. Computer simulations verify that the proposed method provides accurate reconstructions of the principal directions of a diffusion tensor field.

I. Introduction

Magnetic resonance imaging (MRI) has been shown to be effective for imaging diffusion tensor fields through a process known as diffusion-weighted imaging (DWI). In some imaging applications such as cardiac imaging, the primary objective is to use DWI to determine the principal directions of the diffusion tensor field where a complete understanding of the tensor field itself is of secondary interest. It has been established for isolated perfused myocardium that water diffusion anisotropy measured by MRI faithfully parallels histologic anisotropy. In a cardiac study the knowledge of the principal directions of the tensor field provides myocardial fiber organization [1]. Myocardial fiber architecture is a key determinant of the electrical and mechanical properties of the myocardium. On the other hand, the eigenvalues of the diffusion tensor in cardiac tissue may be assumed to be known. These values are similar to diffusivities reported in other human tissues, which are less than half the diffusivity of water at 37^{o} [1].

This paper is concentrated on two tasks. The **first** is presentation of a novel approach of reconstruction of principal directions directly from diffusion-weighted MRI data, assuming known eigenvalues. Because *a priori* information is incorporated, the reconstruction of principal directions requires less measurements than in the case of the reconstruction of a full tensor field. This can be an important asset, because some DWI measurements strongly suffer from systematic errors such as eddy current artifacts [2]. Reducing the number of measurements may be beneficial when it is desired to use only artifact-free measurements. The **second** goal of this paper is to show that a MR tensor tomography diffusion-weighted imaging (TTDWI) approach [4,5] is more effective for reconstructing principal directions than the standard MR DWI techniques.

II. Methods

A. DWI imaging: Tensor tomography versus the conventional MRI technique.

One of the approaches used in DWI is projection reconstruction (PR) imaging [3]. In PR imaging radial lines are acquired in Fourier space instead of rectilinear lines as in 2D Fourier transform imaging methods, for example, echo-planar imaging (EPI). The PR signal during readout can be expressed as

$$s_{\vec{\omega}}(t) = \int\rho(\vec{x})e^{i\gamma\vec{x}\cdot\vec{G}_r t}e^{-\gamma^2 G^2(\vec{\omega}^T D(\vec{x})\vec{\omega})\Lambda^2(\Delta-\Lambda/3)}d\vec{x}\ , \quad (1)$$

where $\vec{G}_r$ is the readout gradient, ρ is the spin density, γ is the gyromagnetic ratio, G is the amplitude of the diffusion weighting gradient, $\vec{\omega}$ is the direction of the applied diffusion weighting gradient, Λ is the length of one lobe of the diffusion pulse, D is the diffusion tensor, and Δ is the separation between each start of the two gradient pulses.

Taking the Fourier transform of $s_{\vec{\omega}}(t)$, (1) can be rewritten as

$$p_{\vec{\omega}\vec{\omega}}(\theta, t) = \int\rho(\vec{x})e^{-g^2(\vec{\omega}^T D(\vec{x})\vec{\omega})}\delta(\vec{x}\cdot\vec{\theta}-t)d\vec{x}\ , \quad (2)$$

where g^2 is a constant and $\vec{\theta}$ is the direction of the readout gradient [4]. According to (2), the PR signal can be presented as a "projection" $p_{\vec{\omega}\vec{\omega}}(\theta, t)$ [which is proportional to the Fourier transform of $s_{\vec{\omega}}(t)$ in (1)] of the two functions ρ and D.

The transition from (1) to (2) is made by applying the well-known Fourier section theorem. Our goal is to reconstruct a 3D diffusion tensor field using fully 3D reconstruction [5]. However, in the following analysis we will restrict our attention to slice-by-slice data acquisition, where the readout direction is defined by the projection angle θ. The necessary information is provided by data sets with different choices of $\vec{\omega}$. This vector has a direction in 3D as shown in Figure 1.

The goal behind DWI is the reconstruction of D, assuming that ρ is a known function. In practice, ρ is reconstructed using (2) when g is set to zero. In the conventional MR DWI technique measurements are made with stationary diffusion gradients, *i. e.* $\vec{\omega}$ is a constant vector. The same function is either projected at every angle θ, as in the case of PR, or acquired line by line in Fourier space, as in the case of EPI. The standard reconstruction technique for PR DWI is the filtered backprojection (FBP) method, for EPI it is the Fourier inversion formula. Exponential terms of D are reconstructed and D is obtained by taking a logarithm. Therefore, conventional DWI provides reconstruction of $\vec{\omega}^T D\vec{\omega}$ for some set of fixed $\vec{\omega}$. For 3D symmetric tensor imaging it is necessary to use six different $\vec{\omega}$ to obtain six different data sets. By appropriate choices of $\vec{\omega}$, conventional MR DWI reconstructs the diffusion tensor components D_{xx}, D_{yy}, etc. from six data sets.

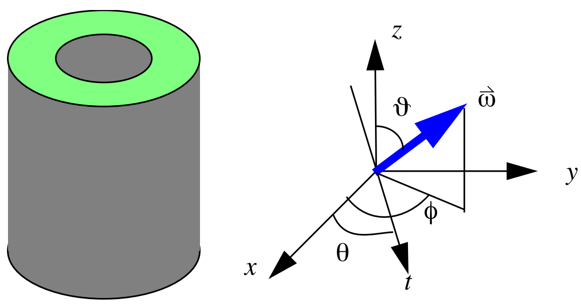

Figure 1. The projection geometry

The tensor tomography DWI (TTDWI) approach is similar to the PR technique but uses the rotating diffusion gradients when $\vec{\omega}$ is a function of θ. In order to reconstruct a 3D tensor field, the six different $\vec{\omega}$ should also be used and all data sets are used simultaneously during the reconstruction process. Because a different function is projected depending on the projection angle, a special reconstruction technique is required [4, 5,6].

B. Parameterization and reconstruction of principal directions when eigenvalues are known

1) 2D case

The main idea can easily be demonstrated in 2D case. The 2D tensor for each pixel can be represented in terms of known eigenvalues and eigenvectors:

$$D_{ij} = \lambda_1 X_i^1 X_j^1 + \lambda_2 X_i^2 X_j^2 \,,\ \lambda_1 > \lambda_2 \,,\ i,j = 1,2 \ . \tag{3}$$

The unknown eigenvectors can be parameterized by the angle Φ, which ensures orthogonality and normalization:

$$\vec{X}^1 = [\cos\Phi, \sin\Phi]^T \,,\ \vec{X}^2 = [-\sin\Phi, \cos\Phi]^T \,,\ 0 \le \Phi < \pi \,. \tag{4}$$

Note that Φ and Φ+π are equivalent, because the principal eigenvectors are defined up to sign. Because the function $\Phi(\vec{x})$ is the unknown variable, one can expect that only a single set of measurements of diffusion-weighted gradients is necessary to reconstruct principal directions.

Conventional DWI provides reconstruction of $\vec{\omega}^T D \vec{\omega}$, where $\vec{\omega} = [\cos\phi, \sin\phi]^T$ at some fixed ϕ. The expression $\vec{\omega}^T D \vec{\omega}$ can be rewritten:

$$\vec{\omega}^T D \vec{\omega} = \lambda_1 \cos(\Phi - \phi)^2 + \lambda_2 \sin(\Phi - \phi)^2 \,. \tag{5}$$

Equation (5) can be solved with respect to Φ:

$$\begin{aligned}
\Phi_1 &= \phi + \operatorname{atan}\left(\sqrt{\frac{\lambda_1 - \vec{\omega}^T D \vec{\omega}}{\vec{\omega}^T D \vec{\omega} - \lambda_2}}\right) + \pi k_1, \ \vec{\omega}^T D \vec{\omega} - \lambda_2 \ne 0 \\
\Phi_2 &= \phi + \operatorname{atan}\left(-\sqrt{\frac{\lambda_1 - \vec{\omega}^T D \vec{\omega}}{\vec{\omega}^T D \vec{\omega} - \lambda_2}}\right) + \pi k_2, \ \vec{\omega}^T D \vec{\omega} - \lambda_2 \ne 0T \\
\Phi_{1,2} &= \phi + \frac{\pi}{2} + \pi k_3, \ \vec{\omega}^T D \vec{\omega} - \lambda_2 = 0,
\end{aligned} \tag{6}$$

where integers k_1, k_2 and k_3 are chosen to ensure that $0 \le \Phi_1, \Phi_2 < \pi$. A single measurement of diffusion-weighted gradients provides two different Φ in general. Therefore, the principal directions are not uniquely defined.

The source of this ambiguity can be seen from other points of view. To define all eigenvalues and eigenvectors it is necessary to know all components of a symmetrical tensor and standard MRI DWI provides reconstruction of them. Suppose we know *a priori* λ_1, λ_2 and reconstruct, for example, D_{xx} (from measurement when ϕ=0). Can we then define D_{yy} and D_{xy}? It is known that two invariants exist for 2D second order tensors:

$$\begin{aligned}
D_{xx} + D_{yy} &= \lambda_1 + \lambda_2 \\
D_{xx}^2 + D_{yy}^2 + 2D_{xy}^2 &= \lambda_1^2 + \lambda_2^2
\end{aligned} \ . \tag{7}$$

Given λ_1, λ_2, and D_{xx} it is clear that the sign of D_{xy} is not defined.

The source of ambiguity can also be presented graphically. Because $(\vec{\omega}^T \vec{X}^1)^2 + (\vec{\omega}^T \vec{X}^2)^2 = 1$, the value of $\vec{\omega}^T D \vec{\omega} = (\lambda_1 - \lambda_2)(\vec{\omega}^T \vec{X}^1)^2 + \lambda_2$ is the same for an equivalent $\left|\vec{\omega}^T \vec{X}^1\right|$. According to Figure 2(a) there are two $\vec{X}^1$ that provide the same $\left|\vec{\omega}^T \vec{X}^1\right|$ in the general case.

We present here an empirical conjecture that the TTDWI method removes this ambiguity. Our results show that the TTDWI method derives a unique estimate of the direction of the principal component in fewer measurements than are required in conventional DWI. It can be shown that only one root in (6) satisfies (5) simultaneously for all ϕ. In the TTDWI method we do not reconstruct $\vec{\omega}^T D \vec{\omega}$, because ϕ is not fixed, but instead we reconstruct D. We can only fit Φ to projection measurements at the angles θ corresponding to one $\vec{\omega}(\phi(\theta))$. However, we should not have duality of Φ, because only one Φ satisfies (5) for all ϕ. In the TTDWI method only projection measurements over θ corresponding to one $\vec{\omega}(\phi(\theta))$ is necessary to define Φ for any given pixel whereas from our above arguments measurements for multiple $\vec{\omega}(\phi)$ are needed for the conventional DWI method. Later we present a simulation that seems to suggest that our conjecture is correct for the case we present in the next section.

2) 3D case, primary anisotropy $\lambda_1 > \lambda_2 = \lambda_3$

This case has a simple graphical interpretation. Only the first principal direction is to be reconstructed. The other two eigenvectors that correspond to the same eigenvalues can be defined arbitrarily in the plane perpendicular to the first principal direction. The tensor components are independent of this particular choice. We can define three eigenvectors for a given voxel using two angles Θ and Φ (similarly ϑ and ϕ are illustrated on Figure 1).

$$\begin{aligned}
\vec{X}^1 &= [\sin\Theta\cos\Phi, \sin\Theta\sin\Phi, \cos\Theta]^T \\
\vec{X}^2 &= [-\sin\Phi, \cos\Phi, 0]^T \\
\vec{X}^3 &= [-\cos\Theta\cos\Phi, -\cos\Theta\sin\Phi, \sin\Theta]^T
\end{aligned} \tag{8}$$

where $0 \le \Phi < 2\pi$ and $0 \le \Theta \le \pi/2$ is only considered. Then $\vec{\omega}$ is defined as $\vec{\omega} = [\sin\vartheta\cos\phi, \sin\vartheta\sin\phi, \cos\vartheta]^T$ and

$$\vec{\omega}^T D \vec{\omega} = \lambda_1 [\sin\vartheta\sin\Theta\cos(\phi - \Phi) + \cos\vartheta\cos\Theta]^2 + \\ \lambda_2 [\sin\vartheta^2 \sin(\phi - \Phi)^2 + (\cos\vartheta\sin\Theta - \sin\vartheta\cos\Theta\cos(\phi - \Phi))^2] \tag{9}$$

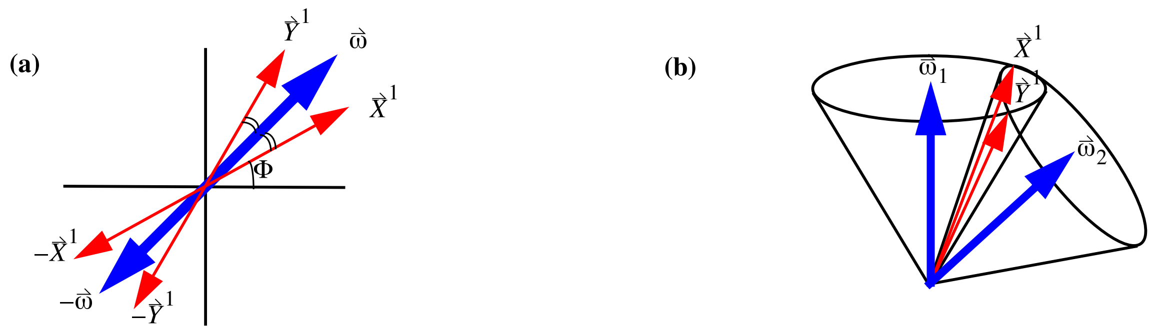

Figure 2. The source of ambiguity in reconstruction of principal directions from data acquired by standard MR DWI when $\vec{\omega}$ is fixed. (a) 2D case. Given $d = \left|\vec{\omega}^T\vec{X}^1\right|$, $\vec{X}^1$ and $\vec{Y}^1$ provide the same d. The principal vectors located in the upper half can be chosen. (b) 3D primary anisotropy case. Two values $d_1 = \left|\vec{\omega}_1^T\vec{X}^1\right|$ and $d_2 = \left|\vec{\omega}_2^T\vec{X}^1\right|$ define two cones. The intersection of these cones in general provides the different principal vector $\vec{Y}^1$, that satisfies d_1 and d_2. The only upper hemisphere is shown.

Because the two functions $\Phi(\vec{x})$ and $\Theta(\vec{x})$ are unknown, one can expect that two measurements with two different $\vec{\omega}$ are necessary when reconstructing the principal directions. This is true in the case of TTDWI, because each $\vec{\omega}$ is a function of θ. However in the case of conventional MR DWI, where $\vec{\omega}$ are fixed, it is necessary to take three measurements to uniquely define the principal directions. Taking into account that $(\vec{\omega}^T\vec{X}^1)^2 + (\vec{\omega}^T\vec{X}^2)^2 + (\vec{\omega}^T\vec{X}^3)^2 = 1$ and $\lambda_2 = \lambda_3$, the equation $\vec{\omega}^T D\vec{\omega} = (\lambda_1 - \lambda_2)(\vec{\omega}^T\vec{X}^1)^2 + \lambda_2$ is satisfied for equal vales of $\left|\vec{\omega}^T\vec{X}^1\right|$. As it can be seen in Figure 2(b), each $\vec{\omega}$ and $\vec{X}^1$ define a cone. Intersection of these two cones defines $\vec{X}^1$ which provides the same $\vec{\omega}_1^T D\vec{\omega}_1$ and $\vec{\omega}_2^T D\vec{\omega}_2$. In general, there are two such $\vec{X}^1$ in the upper hemisphere.

3) 3D case, secondary anisotropy $\lambda_1 > \lambda_2 > \lambda_3$

The existence of secondary myocardial anisotropy in the cross-fiber direction was previously established [1]. In this case it is necessary to estimate all three principal directions. They can be parameterized by three Euler's angles: $\alpha(\vec{x})$, $\beta(\vec{x})$, and $\gamma(\vec{x})$ for each voxel. We omit the corresponding notation here. The three different measurements (three different nonstationary $\vec{\omega}$) may be required to reconstruct these three functions in TTDWI. In the case of a stationary $\vec{\omega}$, that is conventional MR DWI, three measurements are not enough to obtain unique estimates of the principal directions. This can be understood in terms of 3D tensor invariants:

$$Tr(D) = D_{xx} + D_{yy} + D_{zz} = \lambda_1 + \lambda_2 + \lambda_3$$
$$D_{xx}D_{yy} + D_{xx}D_{zz} + D_{yy}D_{zz} - (D_{xy}^2 + D_{xz}^2 + D_{yz}^2) = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3$$
$$det(D) = D_{xx}D_{yy}D_{zz} + 2D_{xy}D_{xz}D_{yz} - (D_{zz}D_{xy}^2 + D_{yy}D_{xz}^2 + D_{xx}D_{yz}^2) = \lambda_1\lambda_2\lambda_3 . \qquad (10)$$

Providing, for example, D_{xx}, D_{yy} and D_{xz}, D_{yz} and D_{xy} are defined up to sign change. Four measurements may be enough. (Measurement components must be chosen carefully. It is a bad choice to measure D_{xx}, D_{yy} and D_{zz} among these four measurements. The tensor estimation still will be ambiguous.)

C. Reconstruction algorithm

For the purposes of this abstract we restricted our attention to the more simple case of primary anisotropy. Only the first principal direction was estimated. Since the MR DWI model in (2) is nonlinear, the reconstruction with nonstationary $\vec{\omega}$ requires use of iterative methods [6,7]. In order to estimate principal directions, the least squares differences between modeled and measured projections can be minimized:

$$L(\Phi(\vec{x}),\Theta(\vec{x})) = \sum_{\theta,t}\sum_{\vec{\omega}}\left\| p_{\vec{\omega}\vec{\omega}}^{model} - p_{\vec{\omega}\vec{\omega}}^{measured}\right\|^2 \qquad (11)$$

where

$$p_{\vec{\omega}\vec{\omega}}^{model}(\theta, t) = \int\rho(\vec{x})e^{-g^2(\vec{\omega}^T D(\vec{x})\vec{\omega})}\delta(\vec{x}\cdot\vec{\theta} - t)d\vec{x} \qquad (12)$$

and

$$\vec{\omega}^T D(\vec{x})\vec{\omega} = \lambda_1[\sin\vartheta\sin\Theta\cos(\phi - \Phi) + \cos\vartheta\cos\Theta]^2 +$$
$$\lambda_2[\sin\vartheta^2\sin(\phi - \Phi)^2 + (\cos\vartheta\sin\Theta - \sin\vartheta\cos\Theta\cos(\phi - \Phi))^2] \qquad (13)$$

In order to minimize (11) with respect to the angle functions, a gradient-type algorithm was applied. Note that the minimization problem is complicated due to the fact that the objective function is periodic with respect to the angle functions; therefore, L has an infinite number of minima. We implemented the gradient descent (GD) algorithm to minimize L. At each iteration this algorithm updates each angle function for a given voxel by its corresponding derivative of L. This algorithm relies on an arbitrarily chosen relaxation parameter ε, that defines the step size in the gradient direction. This parameter should be small enough to not over shoot the downhill direction. The choice of a very small value of ε leads to slow convergence, however, it allows to stay near one of the many minima.

III. Results

A computer generated phantom was used to simulate the diffusion that might be expected in a cardiac study. The phantom is comprised of a circular cylindrical tube. The phantom simulates the mid-ventricular wall of the left ventricle. The spin density ρ is assumed to be uniform inside the phantom and zero outside. The fiber structure of the myocardium is helical. The principal vectors of the diffusion tensor are referenced to a helical fiber structure with material coordinates (X_R, X_F, X_C), which are orthogonal. The fiber axis X_F is located on the plane of the wall normal to the radial axis R. The fiber angle in the circumferential direction has a variation which is continuous and linear. The angle changes from 60^o to -60^o, varying from the endocardial to the epicardial wall in a radial direction. The axis X_C is the cross-fiber in-plane axis and the axis X_R coincides with R. The phantom was chosen to be independent of the z coordinate so that one slice of the reconstruction was enough to represent the entire phantom.

The phantom represented a 32x32 slice image of a cylinder with an inner radius $R_1 = 7$ and an outer radius R_2=14. This grid size was chosen in order to achieve a comprehensive visualization of the vector field of the principal vector. The eigenvalues of the myocardial diffusion tensor were $\lambda_1 = 1.6$, $\lambda_2 = \lambda_3 = 0.7$. The spin density ρ was equal to 1 inside the cylinder and zero outside the cylinder. The parameter g^2 was 0.7, so $g^2\lambda_1 > 1$.

The projection and backprojection operation of the GD algorithm were implemented, using a ray-driven operator. Thirty two projections of the slice were generated over $\theta = [0, \pi)$. The sampling bin width was equal to the reconstructed pixel width. Two projection data sets were used with two fully 3D rotated diffusion gradient directions

$$\vec{\omega}_1 = \left\{ \phi = \theta;\ \vartheta = \begin{matrix} \theta, & \theta \le \pi/2 \\ \pi/2 - \theta, & \theta \ge \pi/2 \end{matrix} \right\} \tag{14}$$

$$\vec{\omega}_2 = \left\{ \phi = \theta + \pi/2;\ \vartheta = \begin{matrix} \theta, & \theta \le \pi/2 \\ \pi/2 - \theta, & \theta \ge \pi/2 \end{matrix} \right\}. \tag{15}$$

The initial condition for the iterative algorithm was uniform: $\Phi = \pi$ and $\Theta = \pi/4$ for every voxel.

Figure 3 shows the behavior of the LS norm in (11) as a function of the iteration number. Because the data is noise-free, the algorithm converged to *L=0*. We chose 4000 iterations for the final reconstruction of the noise-free data. This reconstruction is nearly identical to the original phantom. (The angular difference between phantom and reconstruction principal directions was $1.6^o \pm 1.4^o$.) A smaller iteration number, however, can be used. Figure 4 presents the reconstructed vector field of the first principal vector showing the fiber structure of the heart for one transaxial slice. The first principal direction is well reconstructed in the case of noise-free data from only two diffusion-weighted projection data sets.

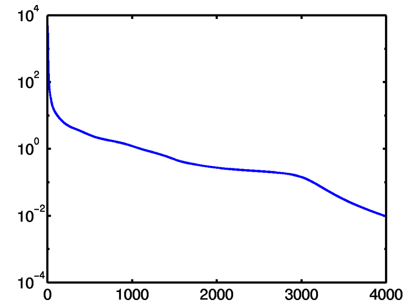

Figure 3. LS norm as a function of the iteration number.

IV. Discussion and Conclusion

We have developed a novel approach to reconstructing the principal directions of the diffusion tensor field. We have also demonstrated that MR TTDWI data acquisition is more efficient than standard MR DWI data acquisition. Further work is needed to prove this result mathematically. Our approach requires use of an iterative algorithm. Further work is required to increase the convergence rate. The choice of an optimal direction of the diffusion gradient direction during acquisition needs further investigation. Acquisition and reconstruction of real data is currently underway.

V. References

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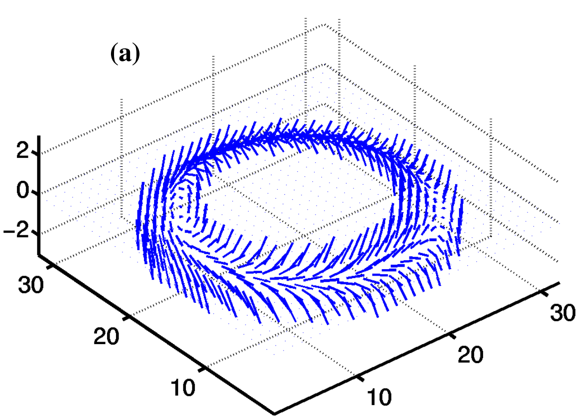

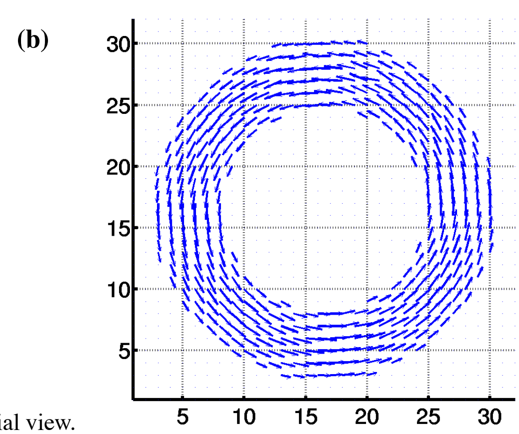

Figure 4. Images of the first principal directions. (a) general view, (b) axial view.